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# A GLOBAL VOLUME LEMMA AND APPLICATIONS

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ABSTRACT

Let $f^t$ be a $C^2$ Axiom A dynamical system on a compact manifold satisfying the transversality condition. We prove that if $B_x(\epsilon,t) = \{y : \mathrm{dist}(f^s x, f^s y) \le \epsilon$ for all $0 \le s \le t\}$, then vol $B_x(\epsilon,t)$ has the order $\exp(\int_0^t \phi(f^s x)\,ds)$ in the continuous time case and $\exp(\sum_{s=0}^{t-1} \phi(f^s x))$ in the discrete time case, where $\phi$ is a Holder continuous extension from basic hyperbolic sets of the negative of the differential expansion coefficient in the unstable direction. An application to the theory of large deviations is given.

## 1. Introduction

Let $M$ be an $m$-dimensional compact connected Riemannian manifold together with a $C^2$ Axiom A dynamical system $f^t$ on it where $t \in (-\infty,\infty)$ (continuous time case) or $t = \ldots,-2,-1,0,1,2,\ldots$ (discrete time case). Suppose that $\Lambda_1,\ldots,\Lambda_\nu$ are the basic hyperbolic sets of $f^t$ on $M$. Then the tangent bundle $TM$ restricted to each $\Lambda_j$ can be written as the Whitney sum of continuous subbundles $T_{\Lambda_j}M = \Gamma^u \oplus \Gamma^{cs}$ where in the discrete time case $\Gamma^{cs} = \Gamma^s$ and in the continuous time case $\Gamma^{cs} = \Gamma^0 \oplus \Gamma^s$ with $\Gamma^0$ being the one-dimensional bundle tangent to the flow $f^t$. This decomposition is invariant with respect to the differential $Df^t$ of $f^t$, and there exist constants $C_1$, $\alpha_1 > 0$ such that

$$\|Df^t\xi\| \le C_1 e^{-\alpha_1 t}\|\xi\| \qquad \text{for } \xi \in \Gamma^s, \quad t \ge 0 \tag{1.1}$$

and

$$\|Df^{-t}\zeta\| \le C_1 e^{-\alpha_1 t}\|\zeta\| \qquad \text{for } \zeta \in \Gamma^u, \quad t \ge 0. \tag{1.2}$$

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Denote by $\mathcal{T}_t(x)$ the Jacobian of the linear map $Df^t : \Gamma^u_x \to \Gamma^u_{f^t x}$ with respect to inner products induced by the Riemannian metric.

Define

$$\phi^u(x) = -\frac{d\mathcal{T}_t(x)}{dt}\bigg|_{t=0} \tag{1.3}$$

in the continuous time case and

$$\phi^u(x) = -\log \mathcal{T}_1(x) \tag{1.4}$$

in the discrete time case. The function $\phi^u$ is defined only on $\bigcup_j \Lambda_j$. We will call a continuous function $\phi$ on $M$ an admissible extension of $\phi^u$ to all of $M$ if $\phi = \phi^u$ on each $\Lambda_j$ and $\phi$ is Holder continuous on a neighborhood of $\bigcup_j \Lambda_j$. We consider the stable and unstable manifolds

$$W^s_x = \{z : \mathrm{dist}(f^t z, f^t x) \to 0 \text{ as } t \to \infty\} \quad \text{and}$$

$$W^u_x = \{z : \mathrm{dist}(f^{-t} z, f^{-t} x) \to 0 \text{ as } t \to \infty\}$$

and the center stable manifold $W^{cs}_x = \bigcup_{t \in \mathbf{R}} f^t(W^s_x)$ for $x \in \bigcup_j \Lambda_j$. We assume that $W^{cs}_x$ and $W^u_y$ are transverse at each point of intersection. This is called the transversality condition (or sometimes the strong transversality condition).

Put also $B_x(\epsilon, t) = \{y : \mathrm{dist}(f^s x, f^s y) \le \epsilon \text{ for all } s \in [0, t]\}$.

THEOREM 1. *Suppose that the transversality condition is satisfied, that $\phi$ is an admissible extension of $\phi^u$ to all of $M$, and that $\epsilon > 0$ is a small positive number. Then, there exists a constant $C_\epsilon > 0$ depending on $\phi$ such that for any $x \in M$ and $t \ge 0$,*

$$C_\epsilon^{-1} \le \mathrm{vol}(B_x(\epsilon, t)) \exp(-S^\phi_t(x)) \le C_\epsilon, \tag{1.5}$$

*where* vol *denotes the Riemannian volume and $S^\phi_t(x) = \sum_{n=0}^{t-1} \phi(f^n x)$ in the discrete time case and $S^\phi_t(x) = \int_0^t \phi(f^u x)\, du$ in the continuous time case.*

This theorem which generalizes the well-known volume lemma from [BR] will be proved in the next section. Two examples at the end of that section imply that, in general, the theorem fails to be true if one drops the transversality condition. In section 3 we will discuss certain applications of Theorem 1.

## 2. Proof of Theorem 1

Since $\bigcup_j \Lambda_j$ contains all limit points of orbits of $f^t$ we have

LEMMA 1. *For any $\delta > 0$ there exists $T(\delta)$ such that for each $x \in M$ there is a $0 \le t(x) \le T(\delta)$ such that $f^{t(x)} x \in \bigcup_j \mathcal{U}_\delta(\Lambda_j)$ where $\mathcal{U}_\delta(K) = \{y : \mathrm{dist}(y, K) < \delta\}$.*

We recall that there is a partial ordering $<$ on basic sets with the property that if $\Lambda_i < \Lambda_j$, then $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$.

Since the transversality condition implies the no cycle property, it follows (see [K2, p. 217]) that

LEMMA 2. *For any $\theta > 0$ small enough there exists a positive $\delta(\theta) < \theta$ such that if for some $x \in M$ and $t > s > 0$ one has $\operatorname{dist}(x, \Lambda_i) \le \delta(\theta)$, $\operatorname{dist}(f^s x, \Lambda_i) \ge \theta$ and $\operatorname{dist}(f^t x, \Lambda_j) \le \delta(\theta)$, then $i \neq j$ and $\Lambda_i < \Lambda_j$; i.e. $\Lambda_j$ comes after $\Lambda_i$ in the filtration of basic hyperbolic sets.*

We will need also the following results proved in [K1], Lemmas 3.1 and 3.4, and in [K2, pp. 133–135].

LEMMA 3. *There exist $C_2, \rho_1 > 0$ such that for any $x \in M$ and $t > 0$ satisfying $\sup_{0 \le u \le t} \operatorname{dist}(f^u x, \Lambda_i) \le \rho$ with $\rho \le \rho_1$ one can find $y \in \Lambda_i$ so that*

$$\sup_{0 \le u \le t} \operatorname{dist}(f^u x, f^u y) \le C_2 \rho. \tag{2.1}$$

LEMMA 4. *There exist $C_3, \rho_2, \alpha_2 > 0$ such that for any $x \in \Lambda_i$ and $y \in B_x(\rho_2, t)$ one has*

$$\operatorname{dist}(f^s y, f^{s+\sigma} x) \le C_3 e^{-\alpha_2 \min(s, t-s)} \max(\operatorname{dist}(x, y), \operatorname{dist}(f^t x, f^t y)) \tag{2.2}$$

*where $\sigma = \sigma(x, y)$ with $|\sigma| \le C_3 \rho_2$ in the continuous time case and $\sigma = 0$ in the discrete time case.*

We now begin the proof of our Theorem.

Take an arbitrary $x \in M$. Define $v_i(x) = \inf\{s : f^s x \in \mathcal{U}_{\delta(\theta)}(\Lambda_i)\}$ and $w_i(x) = \inf\{s > v_i(x) : f^s x \notin \mathcal{U}_\theta(\Lambda_i)\}$ where $\theta$ and $\delta(\theta)$ are as in Lemma 2.

If $\{s : f^s x \in \mathcal{U}_{\delta(\theta)}(\Lambda_i)\} = \emptyset$, then set $v_i(x) = w_i(x) = \infty$.

We may think of $v_i(x)$ (resp. $w_i(x)$) as the first (resp. last) passing time of the orbit of $x$ near $\Lambda_i$.

Let $t > 0$. If for all $i$, $v_i(x) \ge t$, then by Lemma 1, $t \le T(\delta(\theta))$, so (1.5) holds for an appropriate constant $C$. Hence we may assume there is an $i$ with $v_i(x) < t$.

Let $s_i = \dim \Gamma_i^{cs}$, $u_i = \dim \Gamma_i^u$.

By [HPPS], the subbundles $\Gamma_i^{cs}, \Gamma_i^u$ extend to subbundles $\tilde{\Gamma}_i^{cs}, \tilde{\Gamma}_i^u$ on a neighborhood $\mathcal{U}_i$ of $\Lambda_i$ which are locally invariant in the sense that if $f^s z \in \mathcal{U}_i$ for all $s \in [0, t]$, then

$$D_z f^t(\tilde{\Gamma}_{iz}^{cs}) = \tilde{\Gamma}_{if^t z}^{cs} \tag{2.3}$$

and

$$D_z f^t(\tilde{\Gamma}_{iz}^u) = \tilde{\Gamma}_{if^t z}^u. \tag{2.4}$$

Moreover the local stable and unstable manifolds of points in $\Lambda_i$ extend to families $\{\tilde{W}_z^{cs}\}, \{\tilde{W}_z^u\}$ of disks which are also locally invariant and depend continuously on compact parts in the $C^2$ topology. Note that $\dim \tilde{W}_z^{cs} = s_i$ and $\dim \tilde{W}_z^u = u_i$ for $z \in \mathfrak{U}_i$.

Let

$$v_{i_1}(x) < w_{i_1}(x) \leq v_{i_2}(x) < w_{i_2}(x) < \cdots < v_{i_l}(x) < t \tag{2.5}$$

where $l$ is as large as possible. We may and do assume that $\mathfrak{U}_\theta(\Lambda_i) \subset \mathfrak{U}_i$ and $f^{w_i(x)}x \in \mathfrak{U}_i$.

By Lemma 2, $\Lambda_{i_1} < \Lambda_{i_2} < \cdots < \Lambda_{i_l}$; i.e., the orbit of $x$ goes through neighborhoods of a linearly ordered chain in the filtration. We remark also that Axiom A and the transversality condition imply that $u_{i_1} \geq u_{i_2} \geq \cdots \geq u_{i_l}$. By Lemma 1, we have

$$0 \leq t - \sum_{j=1}^{l} (\min(t, w_{i_j}(x)) - v_{i_j}(x)) \leq lT(\delta(\theta)). \tag{2.6}$$

Thus, the total time the orbit of $x$ spends away from $\bigcup_i \mathfrak{U}_{\delta(\theta)}(\Lambda_i)$ is bounded.

In the following, it will simplify notation if we suppress the dependence on $x$ and re-label the subscripts of the $v$'s, $w$'s, $u$'s, $s$'s so that we write $v_j, w_j, u_j, s_j$, for $v_{i_j}(x), w_{i_j}(x), u_{i_j}, s_{i_j}$ with $1 \leq j \leq l$. If $f^t x \in \mathfrak{U}_\theta(\Lambda_{i_l})$, we redefine $w_l$ to be $t$, so that, in any case, we have $t \geq w_l$ and $t - w_l \leq T(\delta(\theta))$ by Lemma 1.

To estimate $\mathrm{vol}(B_x(\epsilon, t))$, we will decompose $B_x(\epsilon, t)$ into smooth foliations of different dimensions, each of which will allow volume estimates in terms of unstable Jacobians. Our decomposition is inspired by the tubular families of [P] and [PS], but our decomposition is much simpler. Similar decompositions into foliations of different dimensions were used by R. C. Robinson in [R] to prove general structural stability theorems.

To motivate the general constructions, we begin by considering a Morse–Smale diffeomorphism of a 3-dimensional manifold $M$ having two saddle fixed points $p_1, p_2$ with $\dim W^u(p_1) = 2$, $\dim W^s(p_2) = 2$ and such that $W^u(p_1)$ has a curve $\gamma$ of transverse intersections with $W^s(p_2)$ as in Fig. 1.

Suppose the eigenvalues of $Df(p_1)$ are $\lambda_1, \lambda_2, \lambda_3$ with $|\lambda_3| < 1 < |\lambda_2| \leq |\lambda_1|$, and those of $Df(p_2)$ are $\lambda_1', \lambda_2', \lambda_3'$ with $|\lambda_3'| \leq |\lambda_2'| < 1 < |\lambda_1'|$. Consider a point $x$ near $W^s(p_1)$ and integers $T_1, T_2 > 0$ such that $f^{T_1}x$ is near $\gamma$, $f^s x$ is near $W^s(p_1) \cup W^u(p_1)$ for $0 \leq s \leq T_1$ and $f^s x$ is near $W^s(p_2) \cup W^u(p_2)$ for $T_1 \leq s \leq T_1 + T_2$. For simplicity, assume $f$ is linear in coordinate systems near $p_1$ and $p_2$.

In the following, we use the notation $A \sim B$ for positive real numbers $A, B$ to mean that there are constants $C_1, C_2$ with $0 < C_1 < AB^{-1} < C_2$. For $\epsilon > 0$ small,

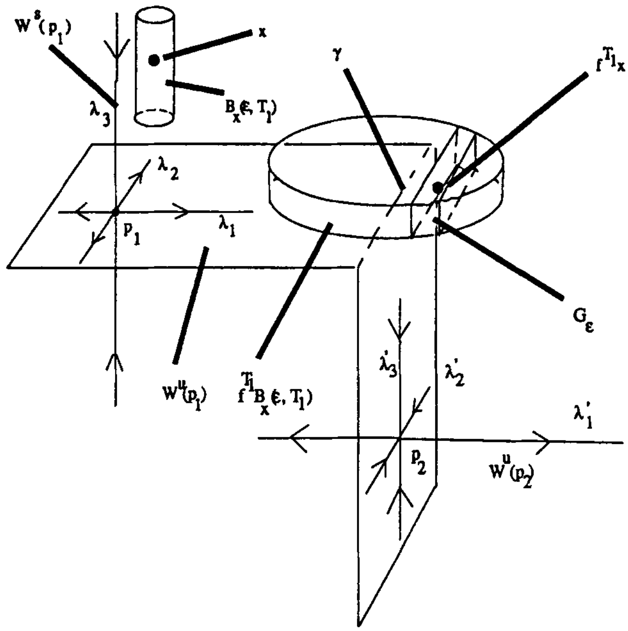


Fig. 1.

$B_x(\epsilon, T_1)$ is a product $B_1^s \times B_1^u$ where $B_1^s$ is a curve near $W^s(p_1)$ of length $\sim\epsilon$ and $B_1^u$ is a 2-disk nearly parallel to $W^u(p_1)$ of area $\sim(\lambda_1\lambda_2)^{-T_1}\epsilon^2$. Further, $f$ contracts in the $\lambda_3$-direction and expands in the $(\lambda_1,\lambda_2)$-direction, so $f^{T_1}(B_x(\epsilon, T_1))$ is a product $\tilde{B}_1^s \times \tilde{B}_1^u$ where

$$\text{length}(\tilde{B}_1^s) \sim \lambda_3^{T_1}\epsilon \quad \text{and} \quad \text{area}(\tilde{B}_1^u) \sim \epsilon^2.$$

Analogously, $B_{f^{T_1}x}(\epsilon, T_2)$ is a product $B_2^s \times B_2^u$ where $B_2^s$ is a 2-disk of area $\sim\epsilon^2$ and $B_2^u$ is a curve of length $\sim(\lambda_1')^{-T_2}\epsilon$. Let $G_\epsilon$ denote the intersection $f^{T_1}(B_x(\epsilon, T_1)) \cap B_{f^{T_1}x}(\epsilon, T_2)$. Thus, $G_\epsilon$ is a parallelotope of size $\sim\epsilon$ in the $\gamma$-direction, of size $\sim(\lambda_1')^{-T_2}\epsilon$ in the $\lambda_1'$-direction, and of size $\sim\lambda_3^{T_1}\epsilon$ in the $\lambda_3$-direction. Clearly, $B_x(\epsilon, T_1 + T_2) = f^{-T_1}G_\epsilon$. In this example, the volume of $B_x(\epsilon, T_1 + T_2)$ is easily estimated in many ways, but we wish to proceed in a manner indicative of the general case.

For $z \in B_x(\epsilon, T_1 + T_2)$, write $z = (z_1, z_2)$ with $z_1 \in B_1^s$ and $z_2 \in B_1^u$. Then, the 2-disk $\{z_1\} \times B_1^u$ is mapped by $f^{T_1}$ to a 2-disk near $W^u(p_1)$. Take a smooth foliation $\mathcal{F}$ by curves of $f^{T_1}(\{z_1\} \times B_1^u)$ each of which is nearly parallel to $W^u(p_2)$. For $\eta \in \mathcal{F}$, the length of $\eta \cap G_\epsilon$ is $\sim(\lambda_1')^{-T_2}\epsilon$. Thus, by Fubini's theorem, the area of $f^{T_1}(\{z_1\} \times B_1^u) \cap G_\epsilon \sim (\lambda_1')^{-T_2}\epsilon$. For each $0 \le s \le T_1$, $f^{-s}(f^{T_1}(\{z_1\} \times B_1^u) \cap G_\epsilon)$

is nearly parallel to $W^u(p_1)$, so each iterate of $f^{-1}$ contracts its area by $(\lambda_1\lambda_2)^{-1}$. Thus,

$$\text{area}(\{z_1\} \times B_1^u \cap f^{-T_1}G_\epsilon) \sim (\lambda_1\lambda_2)^{-T_1}(\lambda_1')^{-T_2}\epsilon.$$

This holds for each $z_1 \in B_1^s$, so again Fubini's theorem gives

$$\text{vol}(B_x(\epsilon, T_1 + T_2)) \sim (\lambda_1\lambda_2)^{-T_1}(\lambda_1')^{-T_2}\epsilon^2 \sim \exp(S^{\phi}_{T_1+T_2}(x))\epsilon^2.$$

We now proceed to the general decomposition of $B_x(\epsilon, t)$.

It will be convenient to make a definition connected with the decomposition we shall give.

Let $\mathfrak{A} \subseteq \prod_{i=1}^l M = \{(z_1, z_2, \ldots, z_l) : z_k \in M\}$ be a subset of the $l$-fold product of $M$ with itself. For $1 \le i \le l$, let $\mathfrak{A}_i$ be the projection of $\mathfrak{A}$ onto the product of the first $i$ factors:

$$\mathfrak{A}_i = \{(z_1, \ldots, z_i) : \text{there are } z_{i+1}, \ldots, z_l \text{ with } (z_1, \ldots, z_i, z_{i+1}, \ldots, z_l) \in \mathfrak{A}\}.$$

Let $k$ be an integer between 1 and $\dim M$. By a $k$-disk we mean a smooth mapping $\gamma$ from the unit $k$-disk $D^k$ in Euclidean space $\mathbf{R}^k$ into $M$. Usually we refer to the image of the $k$-disk $\gamma$ as a $k$-disk also.

Given $x, \epsilon, t, \mathfrak{A}$ as above, an $(x, \epsilon, t, \mathfrak{A})$-disk family is a collection $\{F_{z_1,\ldots,z_k} : 1 \le k \le l\}$ of subsets of $M$ indexed by elements of $\bigcup_{i=1}^l \mathfrak{A}_i$ such that

(1) For fixed $(z_1, \ldots, z_k) \in \mathfrak{A}_k$, $F_{z_1,\ldots,z_k}$ is a $C^1$ $u_k$-disk and $f^{w_k}F_{z_1,\ldots,z_k}$ is $C^1$ near a part of $\tilde{W}^u(f^{w_k}x)$ and has diameter $C\cdot\epsilon$.

(2) For $(z_1, \ldots, z_k) \in \mathfrak{A}_k$, $\{F_{z_1,\ldots,z_k,z} : (z_1, \ldots, z_k, z) \in \mathfrak{A}_{k+1}\}$ is a smooth foliation of $F_{z_1,\ldots,z_k} \cap B_x(\epsilon, w_{k+1})$ (recall that $u_k \le u_{k-1}$).

(3) For each $1 \le k \le l$, there is an $(s_k + u_{k-1} - n)$-disk $D_k \subseteq \tilde{W}^{cs}(f^{v_k}x)$ (which may reduce to a single point) such that $f^{v_k}(F_{z_1,\ldots,z_{k-1},z}) \cap D_k = \{z\}$. If $D_k$ is not a single point, then its diameter is of order $C\cdot\epsilon$.

(4) $\bigcup_{(z_1,\ldots,z_l)\in\mathfrak{A}} F_{z_1,\ldots,z_l} = B_x(\epsilon, w_l)$ and this is a disjoint union.

LEMMA 5. *There is a set $\mathfrak{A} \subseteq \prod_{i=1}^l M$ for which an $(x, \epsilon, t, \mathfrak{A})$-disk family exists.*

We assume Lemma 5 for the moment and proceed to prove Theorem 1. In the following, it will be convenient to denote by $C(\epsilon)$ a function of $\epsilon$, independent of $t$, which is allowed to change in each equation in which it appears. We shall also use the notation $A \sim C(\epsilon)B$ to mean that there are functions $C_1(\epsilon), C_2(\epsilon)$ such that $C_1(\epsilon) < AB^{-1} < C_2(\epsilon)$.

Using the locally invariant families $\{\tilde{W}_z^{cs}\}, \{\tilde{W}_z^u\}$ one can easily extend the proof of the Bowen-Ruelle volume lemma on a hyperbolic set $\Lambda_i$ (see [BR]) to a small neighborhood of $\Lambda_i$. This extended volume lemma which we will call the *local volume lemma* says that (1.5) is true provided $f^s x \in \mathfrak{U}_{\delta(\theta)}(\Lambda_i)$ for all $s \in [0, t]$. In fact, this follows from Lemmas 3 and 4 together with the Bowen–Ruelle volume lemma. Alternatively, one can prove the local volume lemma using the techniques in [PSh].

We will need the following lemma, some variant of which is usually proved as part of the proof of the Bowen–Ruelle volume lemma.

If $D$ is a $u_i$-disk, we use $\mathrm{vol}_i(D)$ to denote its $u_i$-volume.

LEMMA 6. *Suppose $f^s x \in \mathfrak{U}_{\delta(\theta)}(\Lambda_i)$ for all $s \in [0, \beta]$. Let $W$ be a $u_i$-disk $C^2$ near a part of $\tilde{W}^u(f^\beta x)$. Then,*

$$\mathrm{vol}_i(f^{-\beta} W) \sim \exp(S_\beta^\phi(x)) \cdot \mathrm{vol}_i(W).$$

It is this lemma which requires the Holder continuity of $\phi$ on $\mathfrak{U}_\theta(\Lambda_i)$.

Consider the $(x, \epsilon, t, \mathfrak{A})$-disk family $\{F_{z_1,\ldots,z_l} : 1 \le k \le l\}$. The sets $\{F_{z_1,\ldots,z_l} : (z_1, \ldots, z_l) \in \mathfrak{A}\}$ give a decomposition of $B_x(\epsilon, w_l)$ and each $f^{w_l} F_{z_1,\ldots,z_l}$ is a $u_l$-disk near $\tilde{W}^u(f^{w_l} x)$ of diameter $C(\epsilon)$. Let

$$\tilde{F}_{z_1,\ldots,z_l} = f^{w_l} F_{z_1,\ldots,z_l} \cap B_{f^{w_l} x}(\epsilon, t - w_l).$$

Since $t - w_l \le T(\delta(\theta))$, we have that $\tilde{F}_{z_1,\ldots,z_l}$ is also a $u_l$-disk with diameter $C(\epsilon)$ provided $\epsilon$ is small enough depending only on $f$.

Moreover,

$$B_x(\epsilon, t) = B_x(\epsilon, w_l) \cap f^{-w_l} B_{f^{w_l} x}(\epsilon, t - w_l),$$

so

$$f^{w_l} B_x(\epsilon, t) = f^{w_l} B_x(\epsilon, w_l) \cap B_{f^{w_l} x}(\epsilon, t - w_l) = \bigcup_{z_1,\ldots,z_l} \tilde{F}_{z_1,\ldots,z_l}.$$

Thus $\{f^{-w_l} \tilde{F}_{z_1,\ldots,z_l}\}$ gives a decomposition of $B_x(\epsilon, t)$.

We will show that

$$\mathrm{vol}\left(f^{-w_l}\left(\bigcup_{z_1,\ldots,z_l} \tilde{F}_{z_1,\ldots,z_l}\right)\right) \sim \exp(S_t^\phi(x)) \cdot C(\epsilon) \tag{2.7}$$

to prove Theorem 1.

To begin, we have $\mathrm{vol}_l(\tilde{F}_{z_1,\ldots,z_l}) \sim C(\epsilon)$.

By Lemma 6, we see that $f^{v_l - w_l}$ shrinks the $u_l$-volume of each $\tilde{F}_{z_1,\ldots,z_l}$ by about $\exp(S_{w_l - v_l}^\phi(f^{v_l} x))$. Thus,

$$\mathrm{vol}_l(f^{v_l - w_l}(\tilde{F}_{z_1,\ldots,z_l})) \sim \exp(S_{w_l - v_l}^\phi(f^{v_l} x)) \cdot C(\epsilon).$$

Let $\tilde{G}_{z_1,\dots,z_{l-1}} = \bigcup_{z_l \in D_l} f^{v_l - w_l}(\tilde{F}_{z_1,\dots,z_l})$, so that this is a $u_{l-1}$-disk near $f^{v_l}x$. If $D_l$ reduces to the point $\{z_l\}$, then $u_{l-1} = u_l$, and, trivially,

$$\operatorname{vol}_{l-1}(\tilde{G}_{z_1,\dots,z_{l-1}}) \sim \exp(S^{\phi}_{w_l - v_l}(f^{v_l}x)) \cdot C(\epsilon). \tag{2.8}$$

If $D_l$ is a non-trivial disk of diameter $C(\epsilon)$, then Fubini's theorem gives (2.8).

Let $\tilde{F}_{z_1,\dots,z_{l-1}} = f^{w_{l-1} - v_l}\tilde{G}_{z_1,\dots,z_{l-1}}$ so this is a $u_{l-1}$-disk near $f^{w_{l-1}}x$. Since $v_l - w_{l-1} \le T(\delta(\theta))$, we have

$$\operatorname{vol}_{l-1}\Big(\bigcup_{z_l \in D_l} \tilde{F}_{z_1,\dots,z_{l-1}}\Big) \sim \exp(S^{\phi}_{w_l - w_{l-1}}(f^{w_{l-1}}x)) \cdot C(\epsilon).$$

Now, apply similar arguments to the $u_{l-1}$-disks $\tilde{F}_{z_1,\dots,z_{l-1}}$.

Setting $\tilde{G}_{z_1,\dots,z_{l-2}} = \bigcup_{z_{l-1} \in D_{l-1}, z_l \in D_l} f^{v_{l-1} - w_l}(\tilde{F}_{z_1,\dots,z_l})$, we have

$$\begin{aligned}\operatorname{vol}_{l-2}(\tilde{G}_{z_1,\dots,z_{l-2}}) &\sim \exp(S^{\phi}_{w_{l-1} - v_{l-1}}(f^{v_{l-1}}x)) \cdot \exp(S^{\phi}_{w_l - w_{l-1}}(f^{w_{l-1}}x)) \cdot C(\epsilon)\\ &\sim \exp(S^{\phi}_{w_l - v_{l-1}}(f^{v_{l-1}}x)) \cdot C(\epsilon).\end{aligned}$$

Continuing in this manner gives

$$\operatorname{vol}\Big(\bigcup_{z_1,\dots,z_l} f^{v_1 - w_l}(\tilde{F}_{z_1,\dots,z_l})\Big) \sim \exp(S^{\phi}_{w_l - v_1}(f^{v_1}x)) \cdot C(\epsilon).$$

Since $v_1 \le T(\delta(\theta))$ and $t - w_l \le T(\delta(\theta))$, we have

$$\operatorname{vol}(B_x(\epsilon, t)) = \operatorname{vol}\Big(\bigcup_{z_1,\dots,z_l} f^{-w_l}(\tilde{F}_{z_1,\dots,z_l})\Big) \sim \exp(S^{\phi}_{w_l}(x)) \cdot C(\epsilon)$$

and $\exp(S^{\phi}_{w_l}(x)) \sim \exp(S^{\phi}_t(x))$ which gives (2.7). ∎

Now we prove Lemma 5.

We begin with a smooth foliation $\mathcal{F}$ of a small convex neighborhood $\mathcal{V}_1$ of $f^{v_1}x$ into $u_1$-disks $C^1$ near $\tilde{W}^u(f^{v_1}x)$ and $D_1 = \mathcal{V}_1 \cap \tilde{W}^{cs}(f^{v_1}x)$. Let

$$\mathcal{Q}_1 = \{z : \exists F \in \mathcal{F} \text{ with } F \cap D_1 = \{z\}\}.$$

Then, for each $F \in \mathcal{F}$, $f^{w_1 - v_1}(F \cap B_{f^{v_1}x}(\epsilon, w_1 - v_1))$ is a $C^1$ $u_1$-disk of diameter $C(\epsilon)$ near a part of $\tilde{W}^u(f^{w_1}x)$. We let

$$\{F_{z_1}\} = f^{-v_1}\{F \cap B_{f^{v_1}x}(\epsilon, w_1 - v_1) : F \in \mathcal{F}, F \cap D_1 = \{z_1\}\} \cap B_x(\epsilon, v_1).$$

For $z_1 \in D_1$, $f^{v_2}(F_{z_1})$ is a $u_1$-disk $C^1$ near $f^{v_2 - w_1}(\tilde{W}^u(f^{w_1}x))$. Choose points $x_1 \in \Lambda_1$, $x_2 \in \Lambda_2$ such that $f^{w_1}x$ is near $x_1$ and $f^{v_2}x$ is near $x_2$. Continuity of the families $\{\tilde{W}^u\}$ and $\{\tilde{W}^{cs}\}$ gives that $\tilde{W}^u(f^{w_1}x)$ is $C^1$ near $W^u(x_1)$ and $\tilde{W}^{cs}(f^{v_2}x)$ is $C^1$ near $W^{cs}(x_2)$. Since $v_2 - w_1 \le T(\delta(\theta))$, and $f^{v_2 - w_1}(W^u(x_1))$ is transverse to

$W^{cs}(x_2)$, we have that $f^{v_2}(F_{z_1})$ is transverse to $\tilde{W}^{cs}(f^{v_2}x)$. Thus, $D_2 = f^{v_2}(F_{z_1}) \cap \tilde{W}^{cs}(f^{v_2}x) \cap \mathcal{U}_\epsilon(f^{v_2}x)$ is either a point or a smooth $(u_1 + s_2 - n)$-disk of diameter $C(\epsilon)$.

In the latter case, $f^{v_2}(F_{z_1}) \cap \mathcal{U}_\epsilon(f^{v_2}x)$ can be fibered by $u_2$-disks $F'_{z_2}$ which are indexed by $z_2 \in D_2$ and satisfy:

(1) $F'_{z_2} \cap D_2 = \{z_2\}$ and $F'_{z_2}$ is transverse to $D_2$ at $z_2$.

(2) The angle between $F'_{z_2}$ and $\tilde{W}^{cs}(z_2)$ at $z_2$ is bounded below by a constant independent of $z_2$.

(3) The $u_2$-volume of $F'_{z_2}$ is of order $C \cdot \epsilon$.

We let

$$\tilde{F}_{z_1, z_2} = F'_{z_2} \cap B_{f^{v_2}x}(\epsilon, w_2 - v_2).$$

The λ-lemma [P] implies that, for $s \in [0, w_2 - v_2]$, the angle between $f^s(\tilde{F}_{z_1,z_2})$ and $\tilde{W}^{cs}(f^{s+v_2}x)$ remains bounded below by a constant independent of $z_2$, and (assuming that $w_2 - v_2$ is large) $f^{w_2 - v_2}(\tilde{F}_{z_1,z_2})$ is $C^1$ near a part of $\tilde{W}^u(f^{w_2}x)$. Also, since $f$ behaves hyperbolically near each $f^{s+v_2}x$, the $u_2$-volume of $f^{w_2-v_2}(\tilde{F}_{z_1,z_2})$ is of order $C \cdot \epsilon$.

Now, we set $\mathcal{C}_2 = D_1 \times D_2$, and we get $F_{z_1,z_2}$ by pulling back $\tilde{F}_{z_1,z_2}$ to a small neighborhood of $x$ using $f^{-v_2}$ and intersecting with $B_x(\epsilon, v_2)$. That is,

$$F_{z_1,z_2} = f^{-v_2}(\tilde{F}_{z_1,z_2}) \cap B_x(\epsilon, v_2).$$

Continuing in this manner proves Lemma 5. ∎

Remark 1. Using some of the arguments in the proof of Theorem 1 (without computing volumes) one can prove a global shadowing result for systems satisfying Axiom A and the transversality condition. This result says that if $f$ is an Axiom A diffeomorphism on $M$ satisfying the transversality condition, then there exist $C$, $\epsilon_0 > 0$ such that for any $\epsilon$-pseudo-orbit $\{x_i\}$ with $\epsilon \leq \epsilon_0$ there exists $y \in M$ such that $\mathrm{dist}(f^i y, x_i) \leq C\epsilon$. In the case of flows one obtains shadowing with time reparametrization. A slightly weaker version of this shadowing result was stated by Robinson on page 430 in [R].

Remark 2. In general, Theorem 1 fails to hold if the transversality condition is not satisfied. We now give two examples illustrating this.

Example 1. Consider a smooth gradient vector field $X$ on a compact surface $M$ with a saddle connection $\gamma$ between two hyperbolic critical points $p, p'$. Thus if $\eta^t$ is the flow associated to $X$, and $x \in \gamma$, we suppose that $\eta^t(x) \to p$ as $t \to -\infty$ while $\eta^t(x) \to p'$ as $t \to \infty$.

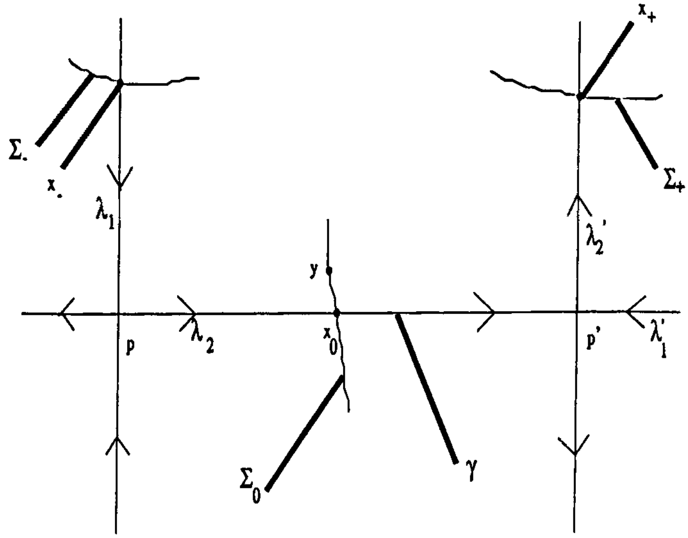


Fig. 2.

Let $\Sigma_0$ be a transverse arc to $\gamma$ at $x_0 \in \gamma$, let $\Sigma_-$ be a transverse arc to $W^s(p)$ at $x_-$, and let $\Sigma_+$ be a transverse arc to $W^u(p')$ at $x_+$ as in Fig. 2. We may and do assume that $\eta^t$ is linear in $C^1$ coordinates near $p$ and $p'$. Let $\lambda_1, \lambda_2$ be the eigenvalues of the derivative of $X$ at $p$, and let $\lambda_1', \lambda_2'$ be those at $p'$ with $\lambda_1 < 0 < \lambda_2$ and $\lambda_1' < 0 < \lambda_2'$.

Consider a point $y$ in $\Sigma_0$ near $x_0$. Let $t_1 = t_1(y)$ be the least positive $t$ such that $\eta^{-t_1}(y) \in \Sigma_-$, and let $t_2 = t_2(y)$ be the least positive $t$ such that $\eta^{t_2}(y) \in \Sigma_+$. Assuming $\operatorname{dist}(x_-,p) \sim \operatorname{dist}(x_+,p') \sim h$, we have

$$\operatorname{dist}(y,x_0) \sim he^{\lambda_1 t_1} \sim he^{-\lambda_2' t_2}.$$

If $\epsilon > 0$ is small, $y$ is close to $x_0$, and $z_1 \equiv \eta^{-t_1}(y)$, then $B_{z_1}(\epsilon,t_1)$ is a small curvilinear rectangle near $x_-$ whose height is $\sim\epsilon$ and whose width is $\sim e^{-\lambda_2 t_1}\cdot\epsilon$. Also, $\eta^{t_1}(B_{z_1}(\epsilon,t_1))$ is a smooth rectangle containing $y$ whose height is $\sim e^{\lambda_1 t_1}\cdot\epsilon$ and width $\sim\epsilon$. Since $e^{\lambda_1 t_1} \sim e^{-\lambda_2' t_2}$, a fixed proportion of $\eta^{t_1}(B_{z_1}(\epsilon,t_1))$ is in $B_y(\epsilon,t_2)$. Thus,

$$\operatorname{vol}(B_{z_1}(\epsilon,t_1+t_2)) \geq C_\epsilon e^{-\lambda_2 t_1}$$

while $S^\phi_{t_1+t_2}(z_1) \sim -\lambda_2 t_1 - \lambda_2' t_2$. Thus, for large $t_1$ and $t_2$, $\operatorname{vol}(B_{z_1}(\epsilon,t_1+t_2))$ is much bigger than $\exp(S^\phi_{t_1+t_2}(z_1))$.

Example 2. In this example of a diffeomorphism in dimension three, there are fixed points $p,p'$ such that:

(1) $\dim W^u(p) = 1$.

(2) $\dim W^s(p') = 1$.

(3) $W^u(p)$ and $W^s(p')$ have a single orbit $O(z)$ of intersections.

(4) $f$ is linear in suitable coordinates near $p$ and near $p'$.

(5) The eigenvalues of $Df(p)$ are real, say $\lambda_1(p), \lambda_2(p), \lambda_3(p)$ and satisfy $\lambda_1(p) > 0$, $\lambda_2(p) < 0$, $\lambda_3(p) < 0$.

(6) The eigenvalues of $Df(p')$ are real, say $\lambda_1(p'), \lambda_2(p'), \lambda_3(p')$ and satisfy $\lambda_1(p') > 0$, $\lambda_2(p') > 0$, $\lambda_3(p') < 0$.

Let $W_1^u$, $W_2^u$ denote $f$-invariant line segments near $p'$ tangent at $p'$ to the eigenspaces of $\lambda_1(p'), \lambda_2(p')$, respectively. We assume that near $z$, $W^u(p)$ contains a line segment which is parallel to $W_1^u$ as in Fig. 3. We also assume that $W_2^u$ is par-

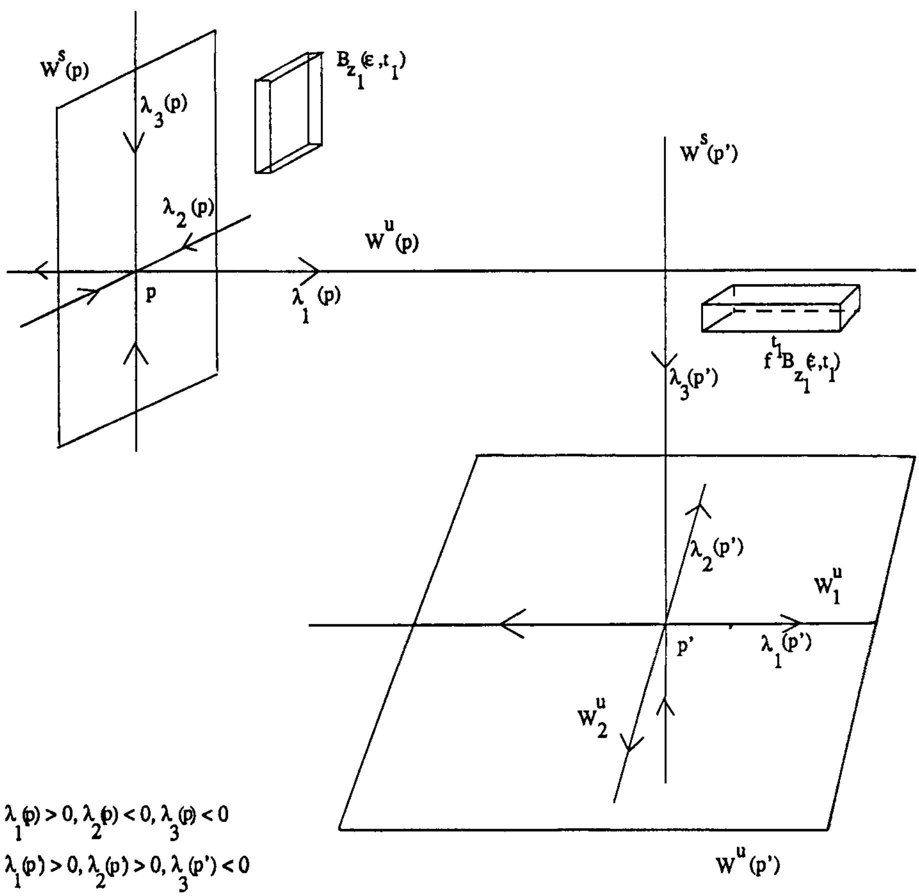


Fig. 3.

allel to the direction of $\lambda_2(p)$. Then, for suitable large $t_1, t_2$ and small $\epsilon$, we have the following.

If $z_1$ is a point whose orbit stays near $W^u(p) \cup W^s(p)$ for time $0 \le t \le t_1$ and stays near $W^u(p') \cup W^s(p')$ for time $t_1 \le t \le t_1 + t_2$, then:

(1) In the direction $W_2^u$, $f^j B_{z_1}(\epsilon, j)$ is less than $\epsilon$ for $t_1 \le j \le t_1 + t_2$.

(2) In the direction of $W_1^u$, $f^{t_1} B_{z_1}(\epsilon, t_1)$ is already of size $\sim\epsilon$, so in this direction

$$B_{z_1}(\epsilon, t_1 + t_2) \sim \epsilon e^{-(t_1\lambda_1(p)+t_2\lambda_1(p'))}.$$

Thus,

$$\operatorname{vol} B_{z_1}(\epsilon, t_1 + t_2) \sim \epsilon^3 e^{-(t_1\lambda_1(p)+t_2\lambda_1(p'))}$$

while

$$S^{\phi}_{t_1+t_2}(z_1) \sim -(\lambda_1(p)t_1 + \lambda_1(p')t_2 + \lambda_2(p')t_2).$$

## 3. Applications

In this section for any set $G \subset M$ let $\bar{G}$ denote its closure and let $G_t = \{x : f^u x \in \bar{G}$ for all $u \in [0, t]\}$. Let $\mathcal{M}_G$ denote the set of all $f^t$-invariant probability measures whose support is contained in $\bar{G}$. For $\mu \in \mathcal{M}_G$, let $h_\mu(f^1)$ denote the entropy of $f^1$ with respect to $\mu$. Let $C(G)$ denote the space of continuous functions on $G$. Let $\epsilon > 0$. A subset $E$ of $G_t$ is called *$(\epsilon, t)$-separated* if whenever $x \ne y \in E$ there is a $j \in [0, t)$ such that $\operatorname{dist}(f^j x, f^j y) > \epsilon$. For $\psi \in C(G)$, let $S_t^\psi(x) = \sum_{n=0}^{t-1} \psi(f^n x)$ in the discrete time case and $S_t^\psi(x) = \int_0^t \psi(f^u x)\, du$ in the continuous time case.

Set

$$Z_G(\epsilon, t, \psi) = \sup\left\{ \sum_{x\in E} \exp(S_t^\psi(x)) : E \subset G_t \text{ is } (\epsilon, t)\text{-}\textit{separated} \right\}.$$

The topological pressure $P_G(\psi)$ is defined to be

$$\lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log Z_G(\epsilon, t, \psi).$$

The variational principle extended to non-invariant sets in Proposition 3.1 of [K3] says

$$P_G(\psi) = \sup_{\mu\in\mathcal{M}_G} \left( \int \psi \, d\mu + h_\mu(f^1) \right) \tag{3.1}$$

where this is defined to be $-\infty$ if $\mathcal{M}_G$ is empty.

Introduce the occupational measures

$$\zeta_x^t = \frac{1}{t}\int_0^t \delta_{f^s x}\, ds$$

in the continuous time case and

$$\zeta_x^t = \frac{1}{t}\sum_{k=0}^{t-1} \delta_{f^k x}$$

in the discrete time case. Remark that $\int_{\bar{G}} \psi\, d\zeta_x^t = S_t^\psi(x)$ for any $x \in G_t$. We denote also by $m$ the Riemannian volume on $M$.

THEOREM 2. *Let $f^t$ be a $C^2$ Axiom A dynamical system on $M$ satisfying the transversality condition, and let $\phi$ be as in Theorem 1. Suppose that $G \subset M$ is an open set such that for each basic hyperbolic set $\Lambda_i$ either $\Lambda_i \subset G$ or $\Lambda_i \cap \bar{G} = \varnothing$. Then for any $V \in C(M)$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \int_{G_t} \exp\left(t \int_{\bar{G}} V d\zeta_x^t\right) dm(x) = P_G(\phi + V). \tag{3.2}$$

*In particular, taking $V \equiv 0$ we obtain*

$$\lim_{t\to\infty} \frac{1}{t} \log \operatorname{vol} G_t = P_G(\phi). \tag{3.3}$$

*Furthermore, for any closed subset $K$ of probability measures on $\bar{G}$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log m\{x : \zeta_x^t \in K\} \le -\inf\{I(\nu) : \nu \in K\} \le P_G(\phi) \tag{3.4}$$

*where*

$$I(\nu) = \begin{cases} -\int \phi\, d\nu - h_\nu(f^1) & \text{if } \nu \in \mathcal{M}_G, \\ \infty & \text{if } \varphi \notin \mathcal{M}_G. \end{cases}$$

PROOF. If $G = M$, then Theorem 1 with Propositions 3.2, 3.3, and Theorem 3.4 from [K3] yield (3.2)–(3.4) directly. Now suppose that $G \subset M$ is a proper open subset satisfying the conditions of the theorem. It suffices to prove (3.2) since (3.3) is a partial case of (3.2), and (3.4) follows from (3.2) by means of the first part of Theorem 2.1 from [K3] together with the variational principle (3.1).

Let $\epsilon > 0$ be small enough so that we can choose a closed set $X$ satisfying:

(1) $X \subset G$.

(2) $\inf_{y \in X} \operatorname{dist}(y, M \setminus G) \geq \epsilon$.

(3) $X$ contains the same basic hyperbolic sets as $G$.

Then

$$\mathcal{M}_X = \mathcal{M}_G. \tag{3.5}$$

Let

$$\gamma_\epsilon(V) = \sup\{|V(y) - V(z)| : y, z \in M \text{ and } \operatorname{dist}(y,z) \leq \epsilon\}.$$

Note that $\gamma_\epsilon(V) \to 0$ as $\epsilon \to 0$.

If $E$ is an $(\epsilon, t)$-separated set and $y, z \in E$, then the sets $B_y(\epsilon/2, t)$ and $B_z(\epsilon/2, t)$ defined before Theorem 1 are disjoint. Moreover, if $X$ contains this $(\epsilon, t)$-separated set $E$ and $y \in E$, then $B_y(\epsilon/2, t) \subset G$. Let $\mathcal{B} = \mathcal{B}(\epsilon, t)$ be the collection of subsets of $X_t$ which are $(\epsilon, t)$-separated.

If we put

$$Q_t(V) = \int_{G_t} \exp\left(t \int_{\bar{G}} V \, d\zeta_x^t\right) dm(x)$$

then, by (1.5),

$$\begin{aligned} Q_t(V) &\geq \sup_{E \in \mathcal{B}} \left\{ \sum_{x \in E} m(B_x(\epsilon/2, t)) \exp\left(t \int_X (V - \gamma_\epsilon(V)) \, d\zeta_x^t\right)\right\} \\ &\geq C^{-1} \sup_{E \in \mathcal{B}} \left\{ \sum_{x \in E} \exp\left(t \int_X (V + \phi - \gamma_\epsilon(V))\right) d\zeta_x^t \right\}. \end{aligned} \tag{3.6}$$

Take $1/t \log$ of both parts of (3.6), let $t \dot{\to} \infty$, and then let $\epsilon \to 0$. Using (3.6) and Proposition 3.1 of [K3], we get

$$\liminf_{t \to \infty} \frac{1}{t} \log Q_t(V) \geq P_X(\phi + V) = P_G(\phi + V). \tag{3.7}$$

If $E$ is a maximal $(\epsilon, t)$-separated set in $G_t$, then $\bigcup_{x \in E} B_x(\epsilon, t) \supset G_t$, so, if $\mathcal{G} = \mathcal{G}(\epsilon, t)$ denotes the collection of subsets $E$ of $G_t$ which are $(\epsilon, t)$-separated, then (1.5) gives

$$\begin{aligned} Q_t(V) &\leq \sup_{E \in \mathcal{G}} \left\{ \sum_{x \in E} m(B_x(\epsilon, t)) \exp\left(t \int_{\bar{G}} (V + \gamma_\epsilon(V)) \, d\zeta_x^t\right)\right\} \\ &\leq C_\epsilon \sup_{E \in \mathcal{G}} \left\{ \sum_{x \in E} \exp\left(t \int_{\bar{G}} (V + \phi + \gamma_\epsilon(V)) \, d\zeta_x^t\right)\right\}. \end{aligned} \tag{3.8}$$

Again, take $1/t \log$ of both parts of 3.8, let $t \to \infty$, and $\epsilon \to 0$. From Proposition 3.1 of [K3], we get

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_t(V) \le P_G(\phi + V).$$

This together with 3.7 gives 3.2 and completes the proof of Theorem 2. ∎

By the variational principle 3.1,

$$P_G(\phi) = \max_i \{P_{\Lambda_i}(\phi) : \Lambda_i \subset G\}.$$

It is known (see [BR]) that $P_{\Lambda_i}(\phi) \le 0$, and $P_{\Lambda_i}(\phi) = 0$ if and only if $\Lambda_i$ is an attractor. Thus, if $\bar{G}$ is disjoint from the attractors, then $P_G(\phi) < 0$ and (3.3) gives a precise escape rate of points from $G$ strengthening the results of [W1] and [W2]. Remark also that (3.4) is the so-called upper large deviation bound for occupational measures. The corresponding lower bound usually will not be true if $G$ contains more than one basic hyperbolic set (see Remark 3.3 in [K3]).

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